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On the Solution of Maximization Problems of Optimal Design by Geometric Programming*

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SUMMARY
A geometric programming method, recently developed for the constrained maximization of posynomials, is presented and illustrated by an application to the optimal design of a torsion bar spring.

1. Introduction

Engineering design problems often involve functions of the form

$$y(x) = \sum_{t=1}^{T} c_t \prod_{n=1}^{N} x_n^{\alpha_{tn}} \tag{1}$$

where the coefficients c_t are positive constants, the exponents α_{tn} are arbitrary real constants, and the design parameters x_n are positive variables, grouped in the vector

$$\boldsymbol{x} = (x_1, \ldots, x_N)\,. \tag{2}$$

Functions of type (1) are called *posynomials*, an abbreviation of positive polynomials.

Geometric programming, abbreviated below by GP, is the mathematical theory developed by Duffin, Peterson and Zener, [1], for the constrained minimization of posynomials, i.e. for solving problems of the type:

Problem 1: Minimize $y_0(\boldsymbol{x})$

s.t. (subject to)

$$x_n > 0 \qquad (n = 1, \ldots, N) \tag{3}$$

$$y_m(\boldsymbol{x}) \leqq 1 \qquad (m = 1, \ldots, M) \tag{4}$$

where the functions $y_m(\boldsymbol{x})$ are posynomials:

$$y_m(\boldsymbol{x}) = \sum_{t=1}^{T_m} c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}} \qquad (m = 0, 1, \ldots, M)\,. \tag{5}$$

Problem 1, called the *primal problem of GP*, is treated in GP via the following related problem, called the *dual problem of GP*.

Problem 2: Maximize the function

$$v(\delta) \equiv \prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}}{\delta_{mt}}\right)^{\delta_{mt}} \prod_{m=1}^{M} \lambda_m^{\lambda_m} \tag{6}$$

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where

$$\boldsymbol{\delta} = (\delta_{01}, \delta_{02}, \ldots, \delta_{0T_0}, \delta_{11}, \ldots, \delta_{1T_1}, \ldots, \delta_{MT_M}) \tag{7}$$

$$\lambda_m = \sum_{t=1}^{T_m} \delta_{mt} \qquad (m = 1, \ldots, M) \tag{8}$$

s.t.

$$\delta_{mt} \geqq 0 \qquad (m = 0, \ldots, M;\ t = 1, \ldots, T_m) \tag{9}$$

$$\sum_{t=1}^{T_0} \delta_{0t} = 1 \tag{10}$$

and

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \alpha_{mtn} \delta_{mt} = 0 \qquad (n = 1, \ldots, N) \tag{11}$$

The correspondence between the primal and the dual problems is such that the term

$$c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}}, \qquad (t = 1, \ldots, T_m),$$

of the posynomial $y_m(x)$, $(m = 0, 1, \ldots, M)$ corresponds to the dual variable δ_{mt}.

The duality theory of GP, [1], gives the useful relations which, under suitable conditions, exist between the primal and the dual problems. In particular, if $\boldsymbol{x}^* = (x_1^*, \ldots, x_N^*)$ is an optimal solution of problem 1, then the vector (7) given by

$$\delta_{mt} = \begin{cases} \dfrac{c_{0t} \prod_{n=1}^{N} (x_n^*)^{\alpha_{0tn}}}{y_0(\boldsymbol{x}^*)} & m = 0;\ t = 1, \ldots, T_0 \\[2ex] \lambda_m c_{mt} \prod_{n=1}^{N} (x_n^*)^{\alpha_{mtn}} & m = 1, \ldots, M;\ t = 1, \ldots, T_m \end{cases} \tag{12}$$

is an optimal solution of the problem 2. It follows from (12), by using (5), (4) and (8), that

$$\delta_{mt} = 0 \qquad (t = 1, \ldots, T_m)$$

for any $m = 1, \ldots, M$ s.t. $y_m(\boldsymbol{x}^*) < 1$.

Conversely, if a vector $\boldsymbol{\delta}^* = (\delta_{01}^*, \ldots, \delta_{MT_M}^*)$ is an optimal solution of problem 2, then each optimal solution $\boldsymbol{x}^*$ of problem 1 satisfies

$$c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}} = \begin{cases} \delta_{0t}^* v(\boldsymbol{\delta}^*) & m = 0;\ t = 1, \ldots, T_0 \\[2ex] \dfrac{\delta_{mt}^*}{\lambda_m^*} \quad \text{for} & m = 1, \ldots, M \text{ s.t. } \lambda_m^* > 0 \\ & t = 1, \ldots, T_m . \end{cases} \tag{13}$$

In either case the optimal values of the primal and dual problems are equal:

$$y_0(\boldsymbol{x}^*) = \min_{\text{s.t. (3), (4)}} y_0(\boldsymbol{x}) = \max_{\text{s.t. (9), (10), (11)}} v(\boldsymbol{\delta}) = v(\boldsymbol{\delta}^*) \tag{14}$$

In GP one solves first the dual problem 2 which is easier to solve since its constraints {(9), (10) and (11)} are linear. An optimal solution $\boldsymbol{\delta}^*$ of the dual problem yields important information about the primal problem 1 even without solving it.

First, the optimal value $\min_{\text{s.t. (3), (4)}} y_0(\boldsymbol{x})$ of the primal problem is, by (14), equal to $v(\boldsymbol{\delta}^*)$.

Second, the terms $c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}}$, $(t = 1, \ldots, T_m)$, of the posynomials $y_m(\boldsymbol{x})$, $(m = 0, 1, \ldots, M)$, are given by (13), and thus it is possible to evaluate the relative importance of these terms without actually solving the primal problem.
Finally, an optimal solution $\boldsymbol{x}$ of the primal problem can be found by solving (13), which is a system of linear equations in the logarithms of x_n, $(n = 1, \ldots, N)$.

These features of GP make it one of the most effective optimization methods presently available to the designer. Some references on applications of GP in engineering design problems are [1] chapter 5, [2] chapter 4, and [3] through [10].

The scope of applications of GP was greatly widened by the recent developments in [11] through [16]. These include extensions and adaptations of GP to optimization problems involving general polynomials rather than posynomials, and general inequalities rather than the one sided constraints (4).

The recent adaptation of GP to the constrained maximization of posynomials, given by the authors in [16], is the subject of this paper. The method is described in section 2, and applied in section 3 to the optimal design of a torsion bar spring ([17], chapter 11) where the energy absorption capability per cycle of repeated loading is maximized, subject to given constraints on the design parameters. Section 4 is a short discussion of the method.

2. The Constrained Maximization of Posynomials

This section gives a heuristic development of some results proved in [15]. We start with the simplest posynomial maximization problem:

Problem 3: Maximize the posynomial

$$y(\boldsymbol{x}) = \sum_{t=1}^{T} c_t \prod_{n=1}^{N} x_n^{\alpha_{tn}} \tag{1}$$

s.t.

$$x_n > 0 \qquad (n = 1, \ldots, N) \tag{3}$$

This problem is equivalent to (in the sense that it shares solutions with) the problem of minimizing the reciprocal of $y(\boldsymbol{x})$.

Problem 4: Minimize $1/y(\boldsymbol{x})$ s.t. (3).

If the posynomial $y(\boldsymbol{x})$ has a single term, i.e. $T = 1$, then its reciprocal $1/y(\boldsymbol{x})$ is also a posynomial and problem 4 is an ordinary unconstrained primal problem of GP.

In the general case we follow the development in [18] and consider an optimal solution $\boldsymbol{x}^*$ of problem 3 and the corresponding maximal value $y^* \equiv y(\boldsymbol{x}^*)$.

The partial derivatives of $1/y$ at $\boldsymbol{x}^*$ satisfy for $k = 1, \ldots, N$

$$\frac{\partial}{\partial x_k^*}\left(\frac{1}{y}\right) = -\frac{1}{(y^*)^2 x_k^*} \sum_{t=1}^{T} \alpha_{tk} c_t \prod_{n=1}^{N} (x_n^*)^{\alpha_{tn}}, \text{ by (1)} \tag{15}$$

$$= 0, \text{ since } \boldsymbol{x}^* \text{ minimizes } 1/y.$$

Let the "weights" δ_t^* be defined by

$$\delta_t^* = \frac{c_t}{y^*} \prod_{n=1}^{N} (x_n^*)^{\alpha_{tn}}. \qquad (t = 1, \ldots, T) \tag{16}$$

These clearly satisfy

$$\delta_t \geqq 0 \qquad (t=1, \ldots, T) \tag{17}$$

$$\sum_{t=1}^{T} \delta_t = 1\,, \quad \text{by (1)}, \tag{18}$$

and by combining (16) and (15)

$$\sum_{t=1}^{T} \alpha_{tk}\, \delta_t = 0 \qquad (k=1, \ldots, N)\,. \tag{19}$$

The minimal $1/y^*$ satisfies

$$\frac{1}{y^*} = \prod_{t=1}^{T} (y^*)^{-\delta_t^*}\,, \quad \text{by (18)},$$

$$= \prod_{t=1} \left(\frac{c_t}{\delta_t^*}\right)^{-\delta_t^*} \prod_{n=1}^{N} (x_n^*)^{-\sum_{t=1}^{T} \alpha_{tn}\delta_t^*}\,, \quad \text{by (16)}$$

$$= \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t^*}\right)^{-\delta_t^*}\,, \quad \text{by (19)}\,. \tag{20}$$

Consider now any variables δ_t, $(t=1, \ldots, T)$, satisfying (17) and (18). The geometric inequality, e.g. [1], implies that for $x_n > 0$ $(n=1, \ldots, N)$

$$y(\boldsymbol{x}) \geqq \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t}\right)^{\delta_t} \prod_{n=1}^{N} x_n^{\sum_{t=1}^{T} \alpha_{tn}\delta_t} \tag{21}$$

with equality if and only if

$$\delta_t = \frac{c_t}{y(\boldsymbol{x})} \prod_{n=1}^{N} x_n^{\alpha_{tn}}\,. \tag{22}$$

If the weights δ_t satisfy (19) in addition to (17) and (18), then the exponents of x_n in (21) vanish and (21) is rewritten as

$$\frac{1}{y(\boldsymbol{x})} \leqq \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t}\right)^{-\delta_t} \quad \text{for any } x_n > 0\,, \qquad (n=1, \ldots, N)\,. \tag{23}$$

Combining (23) and (20) it follows from the minimality of $1/y^*$ that:

$$\prod_{t=1}^{T} \left(\frac{c_t}{\delta_t^*}\right)^{-\delta_t^*} = \frac{1}{y^*} \leqq \frac{1}{y(\boldsymbol{x})} \quad \text{for any} \quad x_n > 0\,, \qquad (n=1, \ldots, N)\,,$$

$$\leqq \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t}\right)^{-\delta_t} \quad \text{for any } \delta_t \qquad (t=1, \ldots, T) \tag{24}$$

satisfying (17), (18) and (19).

Therefore, the vector $\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_T^*)$ defined by (16) is a solution of the following minimization problem:

Problem 5: Minimize the function

$$\prod_{t=1}^{T} \left(\frac{c_t}{\delta_t}\right)^{-\delta_t} \tag{25}^*$$

s.t. (17), (18) and (19).

Problems 4 and 5 are called the *primal* and *dual* problems respectively. Both are minimization problems, a situation different from the duality in GP where the primal problem 1 is a minimi-

* The continuity of (25) for $\delta_t \geqq 0$ $(t=1, \ldots, T)$ requires that $(c_t/\delta_t)^{-\delta_t} = 1$ if $\delta_t = 0$.

zation problem and the dual problem 2 is a maximization problem. The duality relations between the problems 4 and 5 were shown above to include:

(i) The optimal values of the two problems are equal:

$$\min_{\text{s.t. }(3)} \frac{1}{y(\boldsymbol{x})} = \min_{\text{s.t. }(17),(18),(19)} \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t}\right)^{-\delta_t} \tag{26}$$

(ii) If $\boldsymbol{x}^* = (x_1^*, \ldots, x_N^*)$ is an optimal solution of problem 4 then $\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_T^*)$ defined by (16) is an optimal solution of problem 5.

(iii) Conversely, if $\boldsymbol{\delta}^*$ is an optimal solution of problem 5 then, by (22), each optimal solution $\boldsymbol{x}^*$ of problem 4 must satisfy.

$$c_t \prod_{n=1}^{N} x_n^{\alpha_{tn}} = \delta_t^* \prod_{t=1}^{T} \left(\frac{c_t}{\delta_t^*}\right)^{\delta_t^*}, \qquad (t = 1, \ldots, T). \tag{27}$$

Since the positivity constraint (3) is part of the definition of a posynomial, problem 3 can be described as unconstrained maximization of a posynomial. If the constraints (4) are imposed on problem 3 it becomes the following:

Problem 6: Maximize $y_0(\boldsymbol{x})$

s.t.

$$x_n > 0 \qquad (n = 1, \ldots, N) \tag{3}$$

$$y_m(\boldsymbol{x}) \leqq 1 \qquad (m = 1, \ldots, M) \tag{4}$$

where

$$y_m(\boldsymbol{x}) = \sum_{t=1}^{T_m} c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}} \qquad (m = 0, 1, \ldots, M) \tag{5}$$

Taking the reciprocal of $y_0(\boldsymbol{x})$ results in the equivalent minimization problem:

Problem 7: Minimize $1/y_0(\boldsymbol{x})$ s.t. (3) and (4).

For any set of nonnegative variables δ_{0t} $(t = 1, \ldots, T_0)$ satisfying (10), it follows as in (21) that

$$\frac{1}{y_0(\boldsymbol{x})} \leqq \prod_{t=1}^{T_0} \left(\frac{c_{0t}}{\delta_{0t}}\right)^{-\delta_{0t}} \prod_{n=1}^{N} x_n^{-\sum_{t=1}^{T_0} \alpha_{0tn}\delta_{0t}}, \text{ for any } \boldsymbol{x} \tag{28}$$

satisfying (3).

For any such δ_{0t}, $(t = 1, \ldots, T_0)$, we consider the problem of minimizing the right side of (28) s.t. (3), (4) and (5). This problem is called below problem 8 $(\boldsymbol{\delta}_0)$, to denote the fact that the vector

$$\boldsymbol{\delta}_0 = (\delta_{01}, \ldots, \delta_{0T_0}) \tag{29}$$

is a parameter in the definition of the problem.

Problem 8 $(\boldsymbol{\delta}_0)$: Minimize the function

$$\prod_{t=1}^{T_0} \left(\frac{c_{0t}}{\delta_{0t}}\right)^{-\delta_{0t}} \prod_{n=1}^{N} x_n^{-\sum_{t=1}^{T_0} \alpha_{0tn}\delta_{0t}} \tag{30}$$

s.t. (3) and (4),

where $\boldsymbol{\delta}_0 = (\delta_{01}, \ldots, \delta_{0T_0})$ are fixed but arbitrary nonnegative scalars satisfying (10).

The solution of problem 8 $(\boldsymbol{\delta}_0)$ for each $\boldsymbol{\delta}_0$ gives, by (28), an upper bound on the minimal value of $1/y_0(\boldsymbol{x})$ sought in problem 7.

Since (30) is a posynomial (with single term, coefficient $\prod_{t=1}^{T_0}\left(\frac{c_{0t}}{\delta_{0t}}\right)^{-\delta_{0t}}$ and exponents $\sum_{t=1}^{T_0}\alpha_{0tn}\delta_{0t}$), problem 8 $(\boldsymbol{\delta}_0)$ is a primal GP problem, i.e. a special case of problem 1. The dual problem of 8 $(\boldsymbol{\delta}_0)$ is read from problem 2 to be:

Problem 9 $(\boldsymbol{\delta}_0)$: Maximize the function

$$u(\boldsymbol{\delta}_0, \boldsymbol{\delta}) \equiv \prod_{t=1}^{T_0}\left(\frac{c_{0t}}{\delta_{0t}}\right)^{-\delta_{0t}} \prod_{m=1}^{M}\prod_{t=1}^{T_m}\left(\frac{c_{mt}}{\delta_{mt}}\right)^{\delta_{mt}} \lambda_m^{\lambda_m} \tag{31}$$

where

$$\boldsymbol{\delta} = (\delta_{11}, \delta_{12}, \ldots, \delta_{1T_1}, \delta_{21}, \ldots, \delta_{MT_M}) \tag{32}$$

s.t.

$$\delta_{mt} \geqq 0 \qquad (m = 1, \ldots, M;\ t = 1, \ldots, T_m) \tag{33}$$

and

$$-\sum_{t=1}^{T_0}\alpha_{0tn}\delta_{0t} + \sum_{m=1}^{M}\sum_{t=1}^{T_m}\alpha_{mtn}\delta_{mt} = 0 \qquad (n = 1, \ldots, N)\,. \tag{34}$$

Up to this point the parameters $\boldsymbol{\delta}_0$ were arbitrary, but from (28) and problem 7 it is obvious that $\boldsymbol{\delta}_0 = (\delta_{01}, \ldots, \delta_{0T_0})$ must be chosen to minimize the constrained minimum value of (30) which, by the duality theory of (GP), equals the constrained maximum value of (31).

We express this observation in the following:

Problem 10: Minimize the maximum value of $u(\boldsymbol{\delta}_0, \boldsymbol{\delta})$ in problem 9 $(\boldsymbol{\delta}_0)$ s.t.

$$\delta_{0t} \geqq 0 \qquad (t = 1, \ldots, T_0) \tag{35}$$

and

$$\sum_{t=1}^{T_0}\delta_{0t} = 1 \tag{10}$$

Problem 10 is called the *dual* problem, corresponding to the *primal* problem 7. The duality relations between problems 7 and 10, are derived from the duality relations between problems 1 and 2 and between problems 4 and 5. Under suitable conditions, these relations include:

(i) The optimal values of problems 7 and 10 are equal

$$\min_{\text{s.t. (3), (4)}} \frac{1}{y_0(\boldsymbol{x})} = \min_{\substack{\boldsymbol{\delta}_0 \\ \text{s.t. (35), (10)}}} \{ \max_{\substack{\boldsymbol{\delta} \\ \text{s.t. (33), (34)}}} u(\boldsymbol{\delta}_0, \boldsymbol{\delta})\} \tag{36}$$

(ii) If $(\boldsymbol{\delta}_0^*, \boldsymbol{\delta}^*)$ is a minimaximizing solution of problem 10, then each optimal solution $\boldsymbol{x}^*$ of problem 7 must satisfy

$$c_{mt}\prod_{n=1}^{N} x_n^{\alpha_{mtn}} = \begin{cases} \dfrac{\delta_{0t}^*}{u(\boldsymbol{\delta}_0^*, \boldsymbol{\delta}^*)} & m = 0;\ t = 1, \ldots, T_0 \\[2ex] \dfrac{\delta_{mt}^*}{\lambda_m^*} & \text{for } m = 1, \ldots, M \text{ s.t. } \lambda_m^* > 0;\ t = 1, \ldots, T_m \end{cases} \tag{37}$$

These relations suggest the following method of solving problem 7; which consists of the following two steps:

Step 1: Solve problem 10, i.e. find a point $(\boldsymbol{\delta}_0^*, \boldsymbol{\delta}^*)$ minimaximizing $u(\boldsymbol{\delta}_0, \boldsymbol{\delta})$ s.t. (33), (34), (35) and (10).

Important information about the optimal solutions of problem 7 is now available without further work. First, the minimal value of $1/y_0(x)$ in problem 7 is, by (36), equal to $u(\delta_0^*, \delta^*)$. Second, the terms

$$c_{mt} \prod_{n=1}^{N} x_n^{\alpha_{mtn}}, \qquad (t=1, \ldots, T_m),$$

of the posynomials $y_m(x)$, $(m=0, 1, \ldots, M)$, are given by (37).

Step 2: An optimal solution x^* of problem 7 is obtained by solving (37), which is a system of linear equations in the logarithms of x_n $(n=1, \ldots, N)$.

3. An Application to the Optimal Design of a Torsion Bar Spring

The above method is illustrated here by applying it to the optimal design of a torsion bar spring, a problem discussed and solved by R. C. Johnson in [17], chapter 11, using a conventional method.

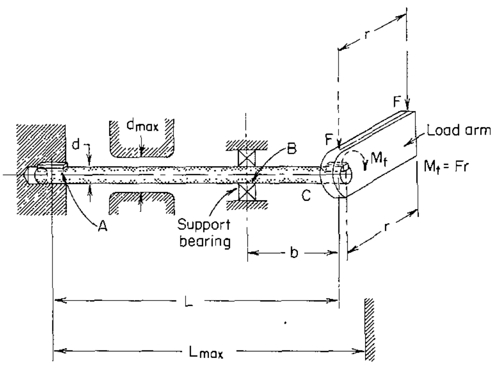

Fig. 1. Torsion bar spring showing practical connection regions and load arm. (Sled-runner type of keyways at both ends.)

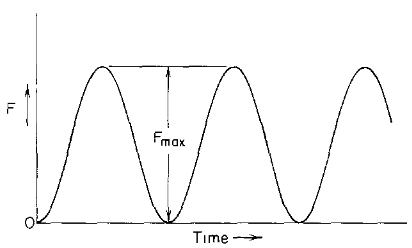

Fig. 2. General variation of force F in figure 1, as introduced in shock loading of torsion bar spring.

The problem is to determine the *material* (to be chosen from a given list of materials) and *geometry* of the torsion bar spring shown in Figure 1, that will maximize the energy absorption capability per cycle of repeated loading.

The notation and assumptions of [17] chapter 11, are adopted here. In particular, the following assumptions are made:

(i) Energy is introduced to the torsion bar spring by a periodic force F (as in Figure 2), applied to the end of the load arm (see Figure 1). The value of $F_{\max}$ is dependent upon the design.

(ii) In calculating the energy storage capability of the torsion bar spring, the following can be neglected:

(a) the energy absorbed in the support bearing at B (see Figure 1)

(b) the energy absorbed in the load arm

(c) the energy absorbed in the torsion bar because of beam shear deflections.

(iii) The diameter d and length L of the torsion bar spring, are bounded from above by d_{max} and L_{max} respectively.
(iv) The stress increase factors, $(K_i)_A$ and $(K_i)_C$, in the bar at sections A and C respectively (see Figure 1) have the common value 1.8.

Solution: Using the analysis of Johnson [17] pp. 366–372, this problem is formulated as the following maximization problem:

Problem 11: Maximize $\text{P.E.} = (\text{P.E.})_t + (\text{P.E.})_b$ where

$$\text{P.E.} = \text{the total energy stored in the torsion bar spring at any instant} \tag{38}$$

$$(\text{P.E.})_t = \text{the energy because of twisting}$$

$$= \frac{32(1+\mu)F^2 r^2 L}{\pi E d^4} \tag{39}$$

$$(\text{P.E.})_b = \text{the energy because of bending} \tag{40}$$

$$= \frac{32F^2 b^2 L}{3\pi E d^4}$$

E = the modulus of elasticity

μ = Poisson's ratio

and the other symbols are defined in Figure 1.
The constraints are:

$$\frac{(1.8)16Fr}{\pi d^3} \leqq \frac{S_e}{(1+p)N} \tag{41}$$

$$\left[\left(\frac{16Fb}{\pi d^3}\right)^2 + \left(\frac{16Fr}{\pi d^3}\right)^2\right]^{\frac{1}{2}} \leqq \frac{S_e}{(1+p)N} \tag{42}$$

$$d \leqq d_{max} \tag{43}$$

$$L \leqq L_{max} \tag{44}$$

and

$$d>0, \quad L>0, \quad r>0, \quad b>0 \text{ and } F>0 . \tag{45}$$

Here

S_e = the published fatigue strength from a standard bending type of fatigue test for the desired number of cycles of life
p = the ratio S_e/S_t, where S_t is the published yield strength of the material from a standard tensile test, and
N = a safety factor, e.g. the discussion in [17] chapter 6.

Obvious transformations (e.g. dividing both sides of (41) by the right side, squaring both sides of (42) and then dividing by the right side etc.) show that problem 11 is of the type called *problem* 6 above. The corresponding problem of type $8(\boldsymbol{\delta}_0)$ is now written as:

Problem $12(\boldsymbol{\delta}_0)$: Minimize the posynomial

$$\frac{\pi}{32} E d^4 F^{-2} L^{-1} \delta_{01}^{\delta_{01}} (3\delta_{02})^{\delta_{02}} (1+\mu)^{\delta_{01}} r^{-2\delta_{01}} b^{-2\delta_{02}} \tag{46}$$

s.t.

$$y_1 \equiv \left[\frac{(1.8)\,16N}{\pi}\left(\frac{1+p}{S_e}\right)\right] F r d^{-3} \leqq 1 \tag{47}$$

$$y_2 \equiv \left[\frac{16^2 N^2}{\pi^2} \left(\frac{1+p}{S_e} \right)^2 \right] (F^2 b^2 d^{-6} + F^2 r^2 d^{-6}) \leqq 1 \tag{48}$$

$$y_3 \equiv \left[\frac{1}{d_{\max}} \right] d \leqq 1 \tag{49}$$

$$y_4 \equiv \left[\frac{1}{L_{\max}} \right] L \leqq 1 \tag{50}$$

and (45).

The parameters $\boldsymbol{\delta}_0 = (\delta_{01}, \delta_{02})$ satisfy

$$\delta_{0t} \geqq 0 \qquad t = 1, 2 \tag{51}$$

$$\delta_{01} + \delta_{02} = 1\,, \tag{52}$$

but are otherwise arbitrary.

For any fixed $\boldsymbol{\delta}_0$, the dual problem of problem $12(\boldsymbol{\delta}_0)$ is found from problem $9(\boldsymbol{\delta}_0)$ to be:

Problem $13(\boldsymbol{\delta}_0)$: Maximize

$$u(\boldsymbol{\delta}_0, \boldsymbol{\delta}) = \left[\frac{\pi}{32} E \delta_{01}^{\delta_{01}} (3\delta_{02})^{\delta_{02}} (1+\mu)^{\delta_{01}} \right] \left(\frac{(1.8)\, 16N}{\pi} \frac{(1+p)}{S_e} \right)^{\delta_1} \times$$

$$\times \left(\frac{16^2 N^2}{\pi^2} \left(\frac{1+p}{S_e} \right)^2 \right)^{\delta_{21}+\delta_{22}} \left(\frac{1}{\delta_{21}} \right)^{\delta_{21}} \left(\frac{1}{\delta_{22}} \right)^{\delta_{22}} \left(\frac{1}{d_{\max}} \right)^{\delta_3} \left(\frac{1}{L_{\max}} \right)^{\delta_4} (\delta_{21}+\delta_{22})^{\delta_{21}+\delta_{22}} \tag{53}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_{21}, \delta_{22}, \delta_3, \delta_4)$

s.t.

$$\begin{array}{lllll}
-2+ & & +\ \delta_1 + 2\delta_{21} + 2\delta_{22} & = 0 & \text{(exponent of } F) \quad (54) \\
-2\delta_{01} & & +\ \delta_1 \qquad\quad + 2\delta_{22} & = 0 & \text{(exponent of } r) \quad (55) \\
 & -2\delta_{02} & \qquad + 2\delta_{21} & = 0 & \text{(exponent of } b) \quad (56) \\
-1 & & \qquad\qquad\qquad\qquad + \delta_4 & = 0 & \text{(exponent of } L) \quad (57) \\
4 & & -3\delta_1 - 6\delta_{21} - 6\delta_{22} + \delta_3 & = 0 & \text{(exponent of } d) \quad (58)
\end{array}$$

and

$$\delta_1 \geqq 0, \quad \delta_{21} \geqq 0, \quad \delta_{22} \geqq 0, \quad \delta_3 \geqq 0, \quad \delta_4 \geqq 0\,. \tag{59}$$

From equations (54) through (58) it follows that

$$\delta_1 = 2(\delta_{01} - \delta_{22}), \quad \delta_{21} = \delta_{02}, \quad \delta_3 = 2 \text{ and } \delta_4 = 1\,. \tag{60}$$

Substituting (60) in (53) and simplifying we get the following problem which is equivalent to problem $13(\boldsymbol{\delta}_0)$.

Problem $14(\boldsymbol{\delta}_0)$: Maximize

$$w(\boldsymbol{\delta}_0, \delta_{22}) = W \delta_{01}^{\delta_{01}} (1.8^2 (1+\mu))^{\delta_{01}} 3^{\delta_{02}} \left(\frac{1}{1.8^2 \delta_{22}} \right)^{\delta_{22}} (\delta_{02} + \delta_{22})^{\delta_{02}+\delta_{22}} \tag{61}$$

s.t.

$$\delta_{22} \leqq \delta_{01} \tag{62}$$

$$\delta_{22} \geqq 0 \tag{63}$$

where

$$W = \left(\frac{\pi E}{32 d_{\max}^2 L_{\max}} \right) \frac{16^2 N^2}{\pi^2} \left(\frac{1+p}{S_e} \right)^2. \tag{64}$$

This is clearly equivalent to the problem of maximizing $\log w(\boldsymbol{\delta}_0, \delta_{22})$ s.t. (62), (63)—whose Lagrangian function is

$$L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = \log w(\boldsymbol{\delta}_0, \delta_{22}) + u_1(\delta_{01} - \delta_{22}) . \tag{65}$$

An application of the Kuhn–Tucker theorem, e.g. [1], gives the following necessary conditions (66)–(70) for δ_{22} to be a solution of the problem 14$(\boldsymbol{\delta}_0)$ with fixed $\boldsymbol{\delta}_0$.

$$\frac{\partial}{\partial \delta_{22}} L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = \frac{1}{w(\boldsymbol{\delta}_0, \delta_{22})} [-(\log 1.8^2 \delta_{22} + 1) + \log(\delta_{02} + \delta_{22}) + 1] - u_1 \leqq 0 \tag{66}$$

$$\delta_{22} \frac{\partial}{\partial \delta_{22}} L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = \left\{ \frac{1}{w(\boldsymbol{\delta}_0, \delta_{22})} [-(\log 1.8^2 \delta_{22} + 1) + \log(\delta_{02} + \delta_{22}) + 1] - u_1 \right\} \delta_{22} = 0 \tag{67}$$

$$\frac{\partial}{\partial u_1} L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = \delta_{01} - \delta_{22} \geqq 0 \tag{68}$$

$$u_1 \frac{\partial}{\partial u_1} L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = (\delta_{01} - \delta_{22}) u_1 = 0 \tag{69}$$

and

$$\delta_{22} \geqq 0 , \quad u_1 \geqq 0 . \tag{70}$$

Let $u_1 = 0$ in (69) and $(\partial/\partial\delta_{22})\, L(\boldsymbol{\delta}_0, \delta_{22}, u_1) = 0$ in (66). These give then the following equation

$$-\log 1.8^2 \delta_{22} + \log(\delta_{02} + \delta_{22}) = 0 \tag{71}$$

yielding

$$\delta_{22}^* = \frac{\delta_{02}}{1.8^2 - 1} \tag{72}$$

which is easily shown to be the optimal solution of problem 14$(\boldsymbol{\delta}_0)$. Substituting δ_{22}^* from (72) in (61), we get

$$w(\boldsymbol{\delta}_0, \delta_{22}^*) = W \delta_{01}^{\delta_{01}} (1.8^2(1+\mu))^{\delta_{01}} 3^{\delta_{02}} \left(\frac{1.8^2 - 1}{1.8^2 \delta_{02}} \right)^{\delta_{02}/(1.8^2 - 1)} \left(\frac{1.8^2 \delta_{02}}{1.8^2 - 1} \right)^{1.8^2 \delta_{02}/(1.8^2 - 1)} . \tag{73}$$

Problem 10 is the dual of the maximization problem 6. From it we write the dual of the maximization problem 11 as follows:

Problem 15: Minimize $w(\boldsymbol{\delta}_0, \delta_{22}^*)$ of (73) s.t. (51) and (52). A similar application of the Kuhn–Tucker theorem gives the optimal solution of problem 15 as:

$$\delta_{01}^* = \frac{3}{3 + (1+\mu)(1.8^2 - 1)} , \quad \delta_{02}^* = \frac{(1+\mu)(1.8^2 - 1)}{3 + (1+\mu)(1.8^2 - 1)} \tag{74}$$

These optimal values of δ_{01}^* and δ_{02}^* give, by (37), the relative contributions of the energy stored because of twisting and the energy stored because of bending, respectively, to the maximal total energy stored in the torsion bar spring, which by (36), is given by $\{w(\delta_{01}^*, \delta_{02}^*, \delta_{22}^*)\}^{-1}$.

The *optimal material* is now determined by calculating $\delta_{01}^*, \delta_{02}^*, \delta_{22}^*$ and $w(\delta_{01}^*, \delta_{02}^*, \delta_{22}^*)$ for each material in the given list, and choosing the material with lowest value of $w(\delta_{01}^*, \delta_{02}^*, \delta_{22}^*)$.

The *optimal geometry* is determined by (37) and the values of $(\boldsymbol{\delta}_0^*, \boldsymbol{\delta}^*)$ from the constraints (41)–(44) as follows:

$$\frac{(1.8)16N}{\pi} \left(\frac{1+p}{S_e} \right) Frd^{-3} = \frac{\delta_1^*}{\delta_1^*} = 1 \tag{75}$$

$$\frac{16^2 N^2}{\pi^2}\left(\frac{1+p}{S_e}\right)^2 F^2 b^2 d^{-6} = \frac{\delta_{21}^*}{\delta_{21}^*+\delta_{22}^*} = \frac{1.8^2-1}{1.8^2} \tag{76}$$

$$\frac{16^2 N^2}{\pi^2}\left(\frac{1+p}{S_e}\right)^2 F^2 r^2 d^{-6} = \frac{\delta_{22}^*}{\delta_{21}^*+\delta_{22}^*} = \frac{1}{1.8^2} \tag{77}$$

$$\frac{d}{d_{\max}} = \frac{\delta_3^*}{\delta_3^*} = 1 \tag{78}$$

$$\frac{L}{L_{\max}} = \frac{\delta_4^*}{\delta_4^*} = 1 \tag{79}$$

giving the optimal values

$$d^* = d_{\max}, \quad L^* = L_{\max}, \quad \frac{b^*}{r^*} = (1.8^2-1)^{\frac{1}{2}}. \tag{80}$$

Good proportions will dictate the actual values of b^* and r^*, but these values must satisfy (80). The optimal solution (80) tallies with that given by Johnson in [17] p. 374.

4. Discussion

As demonstrated in section 3, the method of section 2 for the constrained maximization of posynomials is a powerful analytical and numerical tool in a design situation hitherto treated mostly by *ad hoc* methods.

This method shares with GP the approach and advantages of solving the dual problem first, thus gaining information on the relative importance of the various design parameters and the various terms in the cost functional, and then determining the optimal design parameters (or their logarithms by solving linear equations).

The difference between this method and the usual GP method for constrained minimization of posynomials is that here the dual problem 10 involves minimaximization, rather than the simpler process of maximization involved in the dual problem 2 of GP. There are however two cases where the minimaximization problem 10 is greatly simplified:

Case A. There is a unique solution $(\boldsymbol{\delta}_0, \boldsymbol{\delta})$ of (10) and (34). This occurs when the exponent matrix

$$A = \begin{pmatrix} \alpha_{011} & \alpha_{012} & \ldots\alpha_{01N} \\ \alpha_{021} & \alpha_{022} & \ldots\alpha_{02N} \\ \vdots & & \\ \alpha_{0T_01} & \alpha_{0T_02} & \ldots\alpha_{0T_0N} \\ \alpha_{111} & \alpha_{112} & \ldots\alpha_{11N} \\ \vdots & & \\ \alpha_{MT_M1} & \alpha_{MT_M2} & \alpha_{MT_MN} \end{pmatrix} \tag{81}$$

of dimension $\left(\sum_{m=0}^{M} T_m\right) \times N$ is of rank N and $\sum_{m=0}^{M} T_m = N+1$.

Case B. For any $\boldsymbol{\delta}_0$ there is a unique solution $\boldsymbol{\delta}(\boldsymbol{\delta}_0)$ of (34). This happens if

$$\sum_{m=1}^{M} T_m = N = \text{rank } A.$$

In this case the minimaximization problem 10 reduces to a minimization problem in $\boldsymbol{\delta}_0$.

REFERENCES

[1] R. J. Duffin, E. L. Peterson and C. Zener, *Geometric programming—theory and application*. John Wiley & Sons, Inc., New York (1967) xi + 278 pp.

[2] D. J. Wilde and C. S. Beightler, *Foundations of optimization*, Prentice-Hall, Inc., Englewood Cliffs, N. J., (1967) xiv + 480 pp.

[3] C. Zener, A mathematical aid in optimizing engineering designs, *Proc. Nat. Acad. Sci. U.S.A.*, 47 (1961) 537–539.

[4] C. Zener, A further mathematical aid in optimizing engineering designs, *Proc. Nat. Acad. Sci. U.S.A.*, 48 (1962) 518–522.

[5] C. Zener and R. J. Duffin, Optimization of engineering problems, *Westinghouse Engineer*, 24 (1964) 154–160.

[6] A. Charnes and W. W. Cooper, *Optimizing engineering designs under inequality constraints*, ONR Research Memo No. 64, Northwestern University, Evanston, Illinois 1962.

[7] M. Avriel and D. J. Wilde, Optimal condenser design by geometric programming, *Indust. and Engrg. Chem. Process Design and Development*, 6 (1967) 256–263.

[8] U. Passy and D. J. Wilde, A geometric programming algorithm for solving chemical equilibrium problems, *SIAM J. Appl. Math.*, 16 (1968) 363–373.

[9] U. Passy, Modular design—an exercise in structural geometric programming, *O.R., Statist. and Econ. Mimeo Series No. 26*, Technion-Israel Institute of Technology, Haifa, Israel, September 1968.

[10] M. Heymann and M. Avriel, On a decomposition for a special class of geometric programming problems, *J. Optimiz. Th. Appl.*, 3 (1969) 392–409.

[11] M. Avriel, *Topics in optimization: Block search; applied geometric programming and stochastic geometric programming*, Ph.D. dissertation in Chem. Engrg., Stanford University, Stanford, California (1966).

[12] U. Passy, *Generalizations of geometric programming: Partial control of linear inventory systems*, Ph.D. dissertation in Chem. Engrg., Stanford University, Stanford, California (1966).

[13] U. Passy and D. J. Wilde: Generalized polynomial optimization, *SIAM J. Appl. Math.*, 15 (1967) 1344–1356.

[14] U. Passy, Generalized polynomial optimization, *O.R., Statist. and Econ. Mimeo Series No. 50*, Technion-Israel Institute of Technology, Haifa, Israel, June (1969).

[15] L. D. Pascual and A. Ben-Israel, Constrained maximization of posynomials by geometric programming, *J. Optimiz. Th. Appl.*, 5 (1970) 73–80.

[16] R. J. Duffin, On the concept of linearizing geometric programs, Presented at the meeting of the Operations Research Society of America, Miami, Florida, November 1969.

[17] R. C. Johnson, *Optimum design of mechanical elements*, John Wiley & Sons, Inc., New York (1961) xiv + 535 pp.

[18] C. S. Beightler, R. M. Crisp, Jr. and W. L. Meier, Jr., Optimization by geometric programming, *The Journal of Industrial Engineering*, 19 (1968) 117–120.